Local existence of solutions for the heat equation with a nonlinear boundary condition (Shapes and other properties of the solutions of PDEs)

石毛, 和弘; 佐藤, 龍一

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Local existence of solutions for the heat equation with a nonlinear boundary condition

東北大学大学院理学研究科 石毛和弘 (Kazuhiro Ishige)
Mathematical Institute, Tohoku University

東北大学大学院理学研究科 佐藤龍一 (Ryuichi Sato)
Mathematical Institute, Tohoku University

1 Introduction

This paper is concerned with the heat equation with a nonlinear boundary condition,

$$\begin{cases} \partial_t u = \Delta u, & x \in \Omega,\ t > 0, \\ \nabla u \cdot \nu(x) = |u|^{p-1}u, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = \varphi(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $N \geq 1$, $p > 1$, Ω is a smooth domain in $\mathbf{R}^N$, $\partial_t = \partial/\partial t$ and $\nu = \nu(x)$ is the outer unit normal vector to $\partial\Omega$. For any $\varphi \in BUC(\Omega)$, problem (1.1) has a unique solution

$$u \in C^{2,1}(\Omega \times (0,T]) \cap C^{1,0}(\overline{\Omega} \times (0,T]) \cap BUC(\overline{\Omega} \times [0,T])$$

for some $T > 0$ and the maximal existence time $T(\varphi)$ of the solution can be defined. If $T(\varphi) < \infty$, then

$$\limsup_{t \to T(\varphi)} \|u(t)\|_{L^\infty(\Omega)} = \infty$$

and we call $T(\varphi)$ the blow-up time of the solution u.

Problem (1.1) has been studied in many papers from various points of view (see e.g. [4]–[6], [8]–[12], [14]–[18], [20]–[25], [27], [28], [33], [35] and references therein). In particular, the local well-posedness of the solutions of (1.1) in $L^r(\Omega)$ $(1 \leq r \leq \infty)$ was studied in [4]. See also [6]. However, for problem (1.1), there are few results related to the dependence of the blow-up time on the initial function.

Let $L^r_{uloc,\rho}(\Omega)$ be the uniformly local L^r space in Ω equipped with the norm

$$\|f\|_{r,\rho} := \sup_{x \in \overline{\Omega}} \left(\int_{\Omega \cap B(x,\rho)} |f(y)|^r dy \right)^{1/r},$$

where $1 \leq r < \infty$ and $\rho > 0$. Let $\mathcal{L}^r_{uloc,\rho}(\Omega)$ be the completion of bounded uniformly continuous functions in Ω with respect to the norm $\|\cdot\|_{r,\rho}$, that is,

$$\mathcal{L}^r_{uloc,\rho}(\Omega) := \overline{BUC(\Omega)}^{\|\cdot\|_{r,\rho}}.$$

We set $L^\infty_{uloc,\rho}(\Omega) = L^\infty(\Omega)$ and $\mathcal{L}^\infty_{uloc,\rho}(\Omega) = BUC(\Omega)$. The spaces $L^r_{uloc,\rho}(\Omega)$ and $\mathcal{L}^r_{uloc,\rho}(\Omega)$ are useful for the study of the solutions of parabolic equations in unbounded domains with non-decaying initial functions (see e.g., [7], [31] and references therein).

In this paper we prove the local existence and the uniqueness of the solutions of problem (1.1) with initial functions in $\mathcal{L}^r_{uloc,\rho}(\Omega)$ and obtain the estimates of the blow-up time of the solutions by using the scaling parameter ρ of $\|\varphi\|_{r,\rho}$. The blow-up time of the solution is involved with the degree of the concentration of the initial function, which can be estimated by the scaling parameter ρ of the norm $\|\varphi\|_{r,\rho}$. We give the estimates of the blow-up time by the norm $\|\varphi\|_{r,\rho}$ with a suitable choice of ρ. This also gives a sufficient condition for the existence of global-in-time solutions for problem (1.1) (see Corollary 1.1 and Remark 1.1).

Throughout this paper, following [34, Section 1], we assume that $\Omega \subset \mathbf{R}^N$ is a uniformly regular domain of class C^1. For any $x \in \mathbf{R}^N$ and $\rho > 0$, define

$$B(x,\rho) := \{y \in \mathbf{R}^N : |x-y| < \rho\}, \quad \Omega(x,\rho) := \Omega \cap B(x,\rho), \quad \partial\Omega(x,\rho) := \partial\Omega \cap B(x,\rho).$$

By the trace inequality for $W^{1,1}(\Omega)$-functions and the Gagliardo-Nirenberg inequality we can find $\rho_* \in (0,\infty]$ with the following properties (see Lemma 2.2).

- There exists a positive constant c_1 such that

$$\int_{\partial\Omega(x,\rho)} |v| \, d\sigma \leq c_1 \int_{\Omega(x,\rho)} |\nabla v| \, dy \tag{1.2}$$

  for all $v \in C^1_0(B(x,\rho))$, $x \in \overline{\Omega}$ and $0 < \rho < \rho_*$.

- Let $1 \leq \alpha$, $\beta \leq \infty$ and $\sigma \in [0,1]$ be such that

$$\frac{1}{\alpha} = \sigma\left(\frac{1}{2} - \frac{1}{N}\right) + (1-\sigma)\frac{1}{\beta}. \tag{1.3}$$

  Assume, if $N \geq 2$, that $\alpha \neq \infty$ or $N \neq 2$. Then there exists a constant c_2 such that

$$\|v\|_{L^\alpha(\Omega(x,\rho))} \leq c_2 \|v\|^{1-\sigma}_{L^\beta(\Omega(x,\rho))} \|\nabla v\|^\sigma_{L^2(\Omega(x,\rho))} \tag{1.4}$$

  for all $v \in C^1_0(B(x,\rho))$, $x \in \overline{\Omega}$ and $0 < \rho < \rho_*$.

We remark that, in the case

$$\Omega = \{(x', x_N) \in \mathbf{R}^N \,:\, x_N > \Phi(x')\},$$

where $N \geq 2$ and $\Phi \in C^1(\mathbf{R}^{N-1})$ with $\|\nabla\Phi\|_{L^\infty(\mathbf{R}^{N-1})} < \infty$, (1.2) and (1.4) hold with $\rho_* = \infty$ (see Lemma 2.2). Inequalities (1.2) and (1.4) are used to treat the nonlinear boundary condition.

Next we state the definition of the solution of (1.1).

Definition 1.1 *Let $0 < T \le \infty$ and $1 \le r < \infty$. Let u be a continuous function in $\overline{\Omega} \times (0,T]$. We say that u is a $L^r_{uloc}(\Omega)$-solution of* (1.1) *in $\Omega \times [0,T]$ if*

- $u \in L^\infty(\tau, T : L^\infty(\Omega)) \cap L^2(\tau, T : W^{1,2}(\Omega \cap B(0,R)))$ *for any* $\tau \in (0,T)$ *and* $R > 0$,
- $u \in C([0,T) : L^r_{uloc,\rho}(\Omega))$ *with* $\lim_{t \to 0} \|u(t) - \varphi\|_{r,\rho} = 0$ *for some* $\rho > 0$,
- u *satisfies*

$$\int_0^T \int_\Omega \{-u\partial_t \phi + \nabla u \cdot \nabla \phi\}\, dyds = \int_0^T \int_{\partial\Omega} |u|^{p-1} u \phi \, d\sigma ds \tag{1.5}$$

for all $\phi \in C_0^\infty(\mathbf{R}^N \times (0,T))$.

Here $d\sigma$ is the surface measure on $\partial\Omega$. Furthermore, for any continuous function u in $\overline{\Omega} \times (0,T)$, we say that u is a $L^r_{uloc}(\Omega)$-solution of (1.1) *in $\Omega \times [0,T)$ if u is a $L^r_{uloc}(\Omega)$-solution of* (1.1) *in $\Omega \times [0,\eta]$ for any $\eta \in (0,T)$.*

We remark the following for any ρ, $\rho' \in (0,\infty)$:

- $f \in L^r_{uloc,\rho}(\Omega)$ is equivalent to $f \in L^r_{uloc,\rho'}(\Omega)$;
- $u \in C([0,T] : L^r_{uloc,\rho}(\Omega))$ is equivalent to $u \in C([0,T] : L^r_{uloc,\rho'}(\Omega))$.

These follow from property (i) in Section 2.

Now we are ready to state the main results of this paper. Let $p_* = 1 + 1/N$.

Theorem 1.1 *Let $N \ge 1$ and $\Omega \subset \mathbf{R}^N$ be a uniformly regular domain of class C^1. Let ρ_* satisfy* (1.2) *and* (1.4). *Then, for any $1 \le r < \infty$ with*

$$\begin{cases} r \ge N(p-1) & \text{if} \quad p > p_*, \\ r > 1 & \text{if} \quad p = p_*, \\ r \ge 1 & \text{if} \quad 1 < p < p_*, \end{cases} \tag{1.6}$$

there exists a positive constant γ_1 such that, for any $\varphi \in \mathcal{L}^r_{uloc,\rho}(\Omega)$ with

$$\rho^{\frac{1}{p-1} - \frac{N}{r}} \|\varphi\|_{r,\rho} \le \gamma_1 \tag{1.7}$$

for some $\rho \in (0, \rho_*/2)$, problem (1.1) *possesses a $L^r_{uloc}(\Omega)$-solution u in $\Omega \times [0, \mu\rho^2]$ satisfying*

$$\sup_{0<t<\mu\rho^2} \|u(t)\|_{r,\rho} \le C\|\varphi\|_{r,\rho}, \tag{1.8}$$

$$\sup_{0<t<\mu\rho^2} t^{\frac{N}{2r}} \|u(t)\|_{L^\infty(\Omega)} \le C\|\varphi\|_{r,\rho}. \tag{1.9}$$

Here C and μ are constants depending only on N, Ω, p and r.

Theorem 1.1 implies that $T(\varphi) \ge \mu\rho^2$ under assumption (1.6). Furthermore, we have:

Theorem 1.2 *Assume the same conditions as in Theorem* 1.1. *Let v and w be $L^r_{uloc}(\Omega)$-solutions of* (1.1) *in $\Omega \times [0,T)$ such that $v(x,0) \le w(x,0)$ for almost all $x \in \Omega$, where $T > 0$ and r is as in* (1.6). *Assume, if $r = 1$, that*

$$\limsup_{t\to+0} t^{\frac{1}{2(p-1)}} \left[\|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{L^\infty(\Omega)}\right] < \infty. \tag{1.10}$$

Then there exists a positive constant γ_2 such that, if

$$\rho^{\frac{1}{p-1}-\frac{N}{r}} \left[\|v(0)\|_{r,\rho} + \|w(0)\|_{r,\rho}\right] \le \gamma_2 \tag{1.11}$$

for some $\rho \in (0, \rho_*/2)$, then

$$v(x,t) \le w(x,t) \quad in \quad \Omega \times (0,T).$$

We give some comments related to Theorems 1.1 and 1.2.

(i) Let u be a $L^r_{uloc}(\Omega)$-solution of (1.1) in $\Omega \times [0,T)$. It follows from Definition 1.1 that $u \in L^\infty(\tau, \sigma : L^\infty(\Omega))$ for any $0 < \tau < \sigma < T$. This together with Theorem 6.2 of [12] implies that $u(t) \in BUC(\Omega)$ for any $t \in (0,T)$. This means that $u(0) \in \mathcal{L}^r_{uloc,\rho}(\Omega)$ for any $\rho > 0$.

(ii) Let $1 \le r < \infty$. If, either

$$\text{(a)}\quad f \in L^r_{uloc,1}(\Omega), \quad r > N(p-1) \qquad \text{or} \qquad \text{(b)}\quad f \in L^r(\Omega), \quad r \ge N(p-1),$$

then, for any $\gamma > 0$, we can find a constant $\rho > 0$ such that $\rho^{\frac{1}{p-1}-\frac{N}{r}}\|f\|_{r,\rho} \le \gamma$.

As a corollary of Theorem 1.1, we have:

Corollary 1.1 *Assume the same conditions as in Theorem* 1.1 *and $p > p_*$.*

(i) *For any $\varphi \in L^{N(p-1)}(\Omega)$, problem* (1.1) *has a unique $L^{N(p-1)}_{uloc}(\Omega)$-solution in $\Omega\times[0,T]$ for some $T > 0$.*

(ii) *Assume $\rho_* = \infty$. Then there exists a constant γ such that, if*

$$\|\varphi\|_{L^{N(p-1)}(\Omega)} \le \gamma, \tag{1.12}$$

then problem (1.1) *has a unique $L^{N(p-1)}_{uloc}(\Omega)$-solution u such that*

$$\sup_{0<t<\infty} \|u(t)\|_{L^{N(p-1)}(\Omega)} + \sup_{0<t<\infty} t^{\frac{1}{2(p-1)}}\|u(t)\|_{L^\infty(\Omega)} < \infty.$$

Remark 1.1 *Let $\Omega = \mathbf{R}^N_+ := \{(x', x_N) \in \mathbf{R}^N : x_N > 0\}$. If $1 < p \le p_*$, then problem* (1.1) *possesses no positive global-in-time solutions. See* [11] *and* [18]. *For the case $p > p_*$, it is proved in* [28] *(see also* [27]*) that, if $\varphi \ge 0$, $\varphi \not\equiv 0$ in Ω and*

$$\|\varphi\|_{L^1(\mathbf{R}^N_+)}\|\varphi\|^{N(p-1)-1}_{L^\infty(\mathbf{R}^N_+)} \quad \textit{is sufficiently small,}$$

then there exists a positive global-in-time solution of (1.1). *This also immediately follows from assertion* (ii) *of Corollary* 1.1 *and the comparison principle.*

We explain the idea of the proof of Theorem 1.1. Under the assumptions of Theorem 1.1, there exists a sequence $\{\varphi_n\}_{n=1}^\infty \subset BUC(\Omega)$ such that

$$\lim_{n\to\infty} \|\varphi - \varphi_n\|_{r,\rho} = 0, \qquad \sup_n \|\varphi_n\|_{r,\rho} \le 2\|\varphi\|_{r,\rho}. \tag{1.13}$$

For any $n = 1, 2, \dots$, let u_n satisfy in the classical sense

$$\begin{cases} \partial_t u = \Delta u & \text{in} \quad \Omega \times (0, T_n), \\ \nabla u \cdot \nu(x) = |u|^{p-1}u & \text{on} \quad \partial\Omega \times (0, T_n), \\ u(x, 0) = \varphi_n(x) & \text{in} \quad \Omega, \end{cases} \tag{1.14}$$

where T_n is the blow-up time of the solution u_n. By regularity theorems for parabolic equations (see e.g. [12] and [29, Chapters III and IV]) we see that

$$u_n \in BUC(\overline{\Omega} \times [0, T]), \qquad \nabla u_n \in L^\infty(\Omega \times (\tau, T)), \tag{1.15}$$

for any $0 < \tau < T < T_n$, which imply that u_n is a $L^r_{uloc}(\Omega)$-solution in $\Omega \times [0, T_n)$ for any $1 \le r < \infty$. Set

$$\Psi_{r,\rho}[u_n](t) := \sup_{0\le\tau\le t} \sup_{x\in\overline{\Omega}} \int_{\Omega(x,\rho)} |u_n(y,\tau)|^r \, dy, \qquad 0 \le t < T_n.$$

It follows from (1.7) and (1.13) that

$$\Psi_{r,\rho}[u_n](0)^{\frac{1}{r}} = \|\varphi_n\|_{r,\rho} \le 2\|\varphi\|_{r,\rho} \le 2\gamma_1 \rho^{-\frac{1}{p-1}+\frac{N}{r}}. \tag{1.16}$$

Define

$$\begin{aligned} T_n^* &:= \sup\left\{\sigma \in (0, T_n) \,:\, \Psi_{r,\rho}[u_n](t) \le 6M\Psi_{r,\rho}[u_n](0) \quad \text{in} \quad [0, \sigma]\right\}, \\ T_n^{**} &:= \sup\left\{\sigma \in (0, T_n) \,:\, \rho^{-1} + \|u_n(t)\|_{L^\infty(\Omega)}^{p-1} \le 2t^{-\frac{1}{2}} \quad \text{in} \quad (0, \sigma]\right\}, \end{aligned} \tag{1.17}$$

where M is the integer given in Lemma 2.1. We adapt the arguments in [2], [3] and [26] to obtain uniform estimates of u_n and $u_m - u_n$ with respect to m, $n = 1, 2, \dots$, and prove that

$$\inf_n T_n^* \ge \mu\rho^2, \qquad \inf_n T_n^{**} \ge \mu\rho^2,$$

for some $\mu > 0$. This enables us to prove Theorem 1.1. Theorem 1.2 follows from a similar argument as in Theorem 1.1.

2 Preliminaries

In this section we recall some properties of uniformly local L^r spaces and prove some lemmas related to ρ_*. Furthermore, we give some inequalities used in Section 3. In what follows, the letter C denotes a generic constant independent of $x \in \overline{\Omega}$, n and ρ.

Let $1 \le r < \infty$. We first recall the following properties of $L^r_{uloc,\rho}(\Omega)$:

(i) if $f \in L^r_{uloc,\rho}(\Omega)$ for some $\rho > 0$, then, for any $\rho' > 0$, $f \in L^r_{uloc,\rho'}(\Omega)$ and

$$\|f\|_{r,\rho'} \le C_1 \|f\|_{r,\rho}$$

for some constant C_1 depending only on N, ρ and ρ';

(ii) there exists a constant C_2 depending only on N such that

$$\|f\|_{r,\rho} \le C_2 \rho^{N(\frac{1}{r}-\frac{1}{q})} \|f\|_{q,\rho}, \qquad f \in L^q_{uloc,\rho}(\Omega), \tag{2.1}$$

for any $1 \le r \le q < \infty$ and $\rho > 0$;

(iii) if $f \in L^r(\Omega)$, then $f \in L^r_{uloc,\rho}(\Omega)$ for any $\rho > 0$ and

$$\lim_{\rho \to +0} \|f\|_{r,\rho} = 0. \tag{2.2}$$

Properties (ii) and (iii) are proved by the Hölder inequality and the absolute continuity of $|f|^r\,dy$ with respect to dy. Property (i) follows from the following lemma.

Lemma 2.1 *Let $N \ge 1$ and Ω be a domain in $\mathbf{R}^N$. Then there exists $M \in \{1, 2, \dots\}$ depending only on N such that, for any $x \in \overline{\Omega}$ and $\rho > 0$,*

$$\Omega(x, 2\rho) \subset \bigcup_{k=1}^{n} \Omega(x_k, \rho) \tag{2.3}$$

for some $\{x_k\}_{k=1}^n \subset \overline{\Omega}$ with $n \le M$.

We state a lemma on the existence of ρ_* satisfying (1.2) and (1.4).

Lemma 2.2 *Let $N \ge 1$ and Ω be a uniformly regular domain of class C^1. Then there exists $\rho_* > 0$ such that (1.2) and (1.4) hold. In particular, if*

$$\Omega = \{(x', x_N) \in \mathbf{R}^N \,:\, x_N > \Phi(x')\}, \tag{2.4}$$

where $N \ge 2$ and $\Phi \in C^1(\mathbf{R}^{N-1})$ with $\|\nabla\Phi\|_{L^\infty(\mathbf{R}^{N-1})} < \infty$, then (1.2) and (1.4) hold with $\rho_* = \infty$.

We obtain the following two lemmas by using (1.2) and (1.4).

Lemma 2.3 *Let $N \ge 1$ and $\Omega \subset \mathbf{R}^N$ be a uniformly regular domain of class C^1. Let ρ_* satisfy (1.2) and (1.4). Then there exists a constant C_1 such that*

$$\int_{\partial\Omega(x,\rho)} \phi^2\,d\sigma \le \epsilon \int_{\Omega(x,\rho)} |\nabla\phi|^2\,dy + \frac{C_1}{\epsilon} \int_{\Omega(x,\rho)} \phi^2\,dy \tag{2.5}$$

for all $\phi \in C_0^1(B(x,\rho))$, $\epsilon > 0$, $x \in \overline{\Omega}$ and $\rho \in (0, \rho_*)$. Furthermore, for any $p > 1$ and $r > 0$, there exists a constant C_2 such that

$$\int_{\Omega(x,\rho)} f^{2p+r-2}\,dy \le C_2 \left(\int_{\Omega(x,\rho)} f^{N(p-1)}\,dy \right)^{\frac{2}{N}} \int_{\Omega(x,\rho)} |\nabla f^{\frac{r}{2}}|^2\,dy \tag{2.6}$$

for all nonnegative functions f satisfying $f^{r/2} \in C^1(\Omega(x,\rho))$ with $f = 0$ near $\Omega \cap \partial B(x,\rho)$, $\rho \in (0, \rho_*)$ and $x \in \overline{\Omega}$.

Proof. It follows from (1.4) that

$$\begin{aligned}\int_{\partial\Omega(x,\rho)} \phi^2\, d\sigma &\le C\int_{\Omega(x,\rho)} |\nabla\phi^2|\, dy \le 2C\int_{\Omega(x,\rho)} |\phi|\,|\nabla\phi|\, dy\\ &\le \epsilon\int_{\Omega(x,\rho)} |\nabla\phi|^2\, dy + \frac{C^2}{\epsilon}\int_{\Omega(x,\rho)} \phi^2\, dy\end{aligned}$$

for all $\phi \in W_0^{1,2}(B(x,\rho))$, $\epsilon > 0$, $x \in \overline{\Omega}$ and $\rho \in (0,\rho_*)$. This implies (2.5).

Let $r > 0$ and $0 < \rho < \rho_*$. If $2N(p-1) \ge r$, then, by (1.4) we have

$$\int_{\Omega(x,\rho)} g^{\frac{4}{r}(p-1)+2}\, dy \le C\left(\int_{\Omega(x,\rho)} g^{\frac{2N(p-1)}{r}}\, dy\right)^{\frac{2}{N}} \int_{\Omega(x,\rho)} |\nabla g|^2\, dy \tag{2.7}$$

for all $g \in C_0^1(B(x,\rho))$ and $x \in \overline{\Omega}$. Furthermore, we obtain (2.7) by the Hölder inequality and (1.4) even for the case $2N(p-1) < r$ (see e.g. [32, Lemma 3]). Then, setting $g = f^{r/2}$, we obtain (2.6), and the proof is complete. □

Lemma 2.4 *Assume the same conditions as in Theorem* 1.1. *Let $r \ge 1$, $T > 0$ and f be a nonnegative function such that*

$$f \in C([0,T] : L^r_{uloc,\rho}(\Omega)) \cap L^2(\tau, T : W^{1,2}(\Omega \cap B(0,R)))$$

for any $\rho \in (0,\rho_*/2)$, $\tau \in (0,T)$ and $R > 0$. Let $x \in \overline{\Omega}$ and ζ be a smooth function in $\mathbf{R}^N$ such that

$$\begin{aligned}&0 \le \zeta \le 1 \quad \textit{and} \quad |\nabla\zeta| \le 2\rho^{-1} \quad \textit{in} \quad \mathbf{R}^N,\\ &\zeta = 1 \quad \textit{on} \quad B(x,\rho), \quad \zeta = 0 \quad \textit{outside} \quad B(x,2\rho).\end{aligned}$$

Set $f_\epsilon = f + \epsilon$ for $\epsilon > 0$. Then, for any sufficiently large $k \ge 2$, there exists a constant C such that

$$\begin{aligned}&\sup_{x\in\overline{\Omega}} \int_\tau^t \int_{\partial\Omega(x,2\rho)} f_\epsilon^{p+r-1}\zeta^k\, d\sigma ds\\ &\le C\left[\rho^{\frac{r}{p-1}-N}\Psi_{r,\rho}[f_\epsilon](t)\right]^{\frac{p-1}{r}} \left[\sup_{x\in\overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla f_\epsilon^{\frac{r}{2}}|^2\, dyds + \rho^{-2}(t-\tau)\Psi_{r,\rho}[f_\epsilon](t)\right]\end{aligned} \tag{2.8}$$

for all $0 < \tau < t \le T$, $\rho \in (0,\rho_*/2)$ and $\epsilon > 0$.

Proof. Let $\rho \in (0,\rho_*/2)$. It suffices to consider the case where $\partial\Omega(x,\rho) \ne \emptyset$. Let $k \ge 2$ be such that

$$\frac{k}{2p+r-2}\cdot\frac{r}{2} \ge 1. \tag{2.9}$$

By (1.2) and Lemma 2.1, for any $\delta > 0$, we have

$$
\begin{aligned}
&\int_\tau^t \int_{\partial\Omega(x,2\rho)} f_\epsilon^{p+r-1}\zeta^k \, d\sigma ds \le C \int_\tau^t \int_{\Omega(x,2\rho)} \left|\nabla[f_\epsilon^{p+r-1}\zeta^k]\right| \, dyds \\
&\le C \int_\tau^t \int_{\Omega(x,2\rho)} f_\epsilon^{p+\frac{r}{2}-1}|\nabla f_\epsilon^{\frac{r}{2}}|\zeta^k \, dyds + C \int_\tau^t \int_{\Omega(x,2\rho)} f_\epsilon^{p+r-1}|\nabla\zeta|\zeta^{k-1} \, dyds \\
&\le C\delta \int_\tau^t \int_{\Omega(x,2\rho)} f_\epsilon^{2p+r-2}\zeta^k \, dyds \\
&\qquad + C\delta^{-1} \int_\tau^t \int_{\Omega(x,2\rho)} |\nabla f_\epsilon^{\frac{r}{2}}|^2\zeta^k \, dyds + C\delta^{-1} \int_\tau^t \int_{\Omega(x,2\rho)} f_\epsilon^r \zeta^{k-2}|\nabla\zeta|^2 \, dy\, ds \\
&\le C\delta \int_\tau^t \int_{\Omega(x,2\rho)} f_\epsilon^{2p+r-2}\zeta^k \, dyds \\
&\qquad + C\delta^{-1} \sup_{x\in\overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla f_\epsilon^{\frac{r}{2}}|^2 \, dyds + C\delta^{-1}\rho^{-2}(t-\tau)\Psi_{r,\rho}[f_\epsilon](t)
\end{aligned}
\tag{2.10}
$$

for $0 < \tau < t \le T$, where C is a constant independent of ϵ and δ. Set $g_\epsilon := f_\epsilon\zeta^{k/(2p+r-2)}$. It follows from (2.9) that $f_\epsilon^{r/2} = 0$ near $\Omega \cap \partial B(x, 2\rho)$. Then, by Lemmas 2.1 and 2.3 we have

$$
\begin{aligned}
&\int_\tau^t \int_{\Omega(x,2\rho)} f_\epsilon(y,\tau)^{2p+r-2}\zeta^k \, dyds = \int_\tau^t \int_{\Omega(x,2\rho)} g_\epsilon(y,\tau)^{2p+r-2} \, dyds \\
&\le C \sup_{0<s<t} \left(\int_{\Omega(x,2\rho)} g_\epsilon(y,s)^{N(p-1)} \, dy \right)^{\frac{2}{N}} \int_\tau^t \int_{\Omega(x,2\rho)} |\nabla g_\epsilon^{\frac{r}{2}}|^2 \, dyds \\
&\le C \sup_{0<s<t} \left(\rho^{\frac{r}{p-1}-N} \int_{\Omega(x,2\rho)} f_\epsilon(y,s)^r \, dy \right)^{\frac{2(p-1)}{r}} \\
&\qquad\qquad \times \left[\int_\tau^t \int_{\Omega(x,2\rho)} |\nabla f_\epsilon^{\frac{r}{2}}|^2 \, dyds + \rho^{-2} \int_\tau^t \int_{\Omega(x,2\rho)} f_\epsilon^r \, dyds \right] \\
&\le C \left[\rho^{\frac{r}{p-1}-N} \Psi_{r,\rho}[f_\epsilon](t) \right]^{\frac{2(p-1)}{r}} \\
&\qquad\qquad \times \left[\sup_{x\in\overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla f_\epsilon^{\frac{r}{2}}|^2 \, dyds + \rho^{-2}(t-\tau)\Psi_{r,\rho}[f_\epsilon](t) \right]
\end{aligned}
\tag{2.11}
$$

for $0 < \tau < t \le T$. Therefore, taking $\delta = [\rho^{\frac{r}{p-1}-N}\Psi_{r,\rho}[f_\epsilon](t)]^{-(p-1)/r}$, by (2.10) and (2.11) we obtain (2.8), and the proof is complete. □

3 Proof of Theorems 1.1 and 1.2 in the case $r > 1$.

Let v and w be $L^r_{uloc}(\Omega)$-solutions of (1.1) in $\Omega \times [0, T]$, where $0 < T < \infty$ and r is as in (1.6). Set $z := v - w$ and $z_\epsilon := \max\{z, 0\} + \epsilon$ for $\epsilon \ge 0$. Then z_ϵ satisfies

$$
\partial_t z_\epsilon \le \Delta z_\epsilon \quad \text{in} \quad \Omega \times (0, T], \qquad \nabla z_\epsilon \cdot \nu(x) \le a(x,t) z_\epsilon \quad \text{on} \quad \partial\Omega \times (0, T], \tag{3.1}
$$

in the weak sense (see e.g. [13, Chapter II]). Here

$$a(x,t) := \begin{cases} \dfrac{|v(x,t)|^{p-1}v(x,t) - |w(x,t)|^{p-1}w(x,t)}{v(x,t)-w(x,t)} & \text{if} \quad v(x,t) \neq w(x,t), \\ p|v(x,t)|^{p-1} & \text{if} \quad v(x,t) = w(x,t), \end{cases} \tag{3.2}$$

which satisfies

$$0 \le a(x,t) \le C(|v|^{p-1} + |w|^{p-1}) \quad \text{in} \quad \Omega \times (0,T]. \tag{3.3}$$

In this section we give some estimates of z, and prove Theorems 1.1 and 1.2 in the case $r > 1$.

We first give an L^∞_{loc} estimate of z_0 by using the Moser iteration method with the aid of (1.17). For related results, see [17].

Lemma 3.1 *Assume the same conditions as in Theorem* 1.1. *Let v and w be $L^r_{uloc}(\Omega)$-solutions of* (1.1) *in $\Omega \times [0,T]$, where $0 < T < \infty$ and $r \ge 1$. Set $z_0 := \max\{v-w, 0\}$ and $a = a(x,t)$ as in* (3.2). *Then there exists a constant C such that*

$$\|z_0(t)\|_{L^\infty(\Omega(x,R_1)\times(t_1,t))} \le CD^{\frac{N+2}{2r}} \left(\int_{t_2}^t \int_{\Omega(x,R_2)} z_0^r \, dyds \right)^{1/r}, \tag{3.4}$$

$$\int_{t_1}^t \int_{\Omega(x,R_1)} |\nabla z_0|^2 \, dyds \le CD \int_{t_2}^t \int_{\Omega(x,R_2)} z_0^2 \, dyds, \tag{3.5}$$

for all $x \in \overline{\Omega}$, $0 < R_1 < R_2 < \rho_*$ and $0 < t_2 < t_1 < t \le T$, where

$$D := \|a\|^2_{L^\infty(\partial\Omega(x,R_2)\times(t_2,t))} + (R_2 - R_1)^{-2} + (t_1 - t_2)^{-1}.$$

Proof. Let $x \in \overline{\Omega}$, $0 < R_1 < R_2 < \rho_*$ and $0 < t_2 < t_1 < t \le T$. For $j = 0, 1, 2, \dots$, set

$$r_j := R_1 + (R_2 - R_1)2^{-j}, \quad \tau_j := t_1 - (t_1 - t_2)2^{-j}, \quad Q_j := \Omega(x, r_j) \times (\tau_j, t).$$

Let ζ_j be a piecewise smooth function in Q_j such that

$$\begin{aligned} & 0 \le \zeta_j \le 1 \quad \text{in} \quad \mathbf{R}^N, \quad \zeta_j = 1 \quad \text{on} \quad Q_{j+1}, \\ & \zeta_j = 0 \quad \text{near} \quad \partial\Omega(x,r_j) \times [\tau_j, t] \cup \Omega(x,r_j) \times \{\tau_j\}, \\ & |\nabla \zeta_j| \le \frac{2^{j+1}}{R_2 - R_1} \quad \text{and} \quad 0 \le \partial_t \zeta_j \le \frac{2^{j+1}}{t_1 - t_2} \quad \text{in} \quad Q_j. \end{aligned} \tag{3.6}$$

Let $\alpha_0 > 1$ and $\epsilon > 0$. For any $\alpha \ge \alpha_0$, multiplying (3.1) by $z_\epsilon^{\alpha-1}\zeta_j^2$ and integrating it on Q_j, we obtain

$$\begin{aligned} & \frac{1}{\alpha} \sup_{\tau_j < s < t} \int_{\Omega(x,r_j)} z_\epsilon^\alpha \zeta_j^2 \, dy + \frac{\alpha - 1}{2} \iint_{Q_j} z_\epsilon^{\alpha-2} |\nabla z_\epsilon|^2 \zeta_j^2 \, dyds \\ & \le \frac{4}{\alpha} \iint_{Q_j} z_\epsilon^\alpha \zeta_j |\partial_t \zeta_j| \, dyds + \frac{4}{\alpha - 1} \iint_{Q_j} z_\epsilon^\alpha |\nabla \zeta_j|^2 \, dyds \\ & \qquad + 2 \int_{\tau_j}^t \int_{\partial\Omega(x,r_j)} a(y,s) z_\epsilon^\alpha \zeta_j^2 \, d\sigma ds. \end{aligned} \tag{3.7}$$

This calculation is somewhat formal, however it is justified by the same argument as in [29, Chapter III] (see also [13]). Then it follows that

$$
\begin{aligned}
\sup_{\tau_j<s<t}\int_{\Omega(x,r_j)} z_\epsilon^\alpha\zeta_j^2\,dy+\iint_{Q_j}|\nabla[z_\epsilon^{\frac{\alpha}{2}}\zeta_j]|^2\,dyds &\le C\iint_{Q_j} z_\epsilon^\alpha\zeta_j\partial_t\zeta_j\,dyds\\
&+C\iint_{Q_j} z_\epsilon^\alpha|\nabla\zeta_j|^2\,dyds+C\alpha\int_{\tau_j}^t\int_{\partial\Omega(x,r_j)} a(y,s)z_\epsilon^\alpha\zeta_j^2\,d\sigma ds
\end{aligned}
\tag{3.8}
$$

for all $j=0,1,2,\dots$ and $\alpha\ge\alpha_0$. On the other hand, by Lemma 2.3 we have

$$
\begin{aligned}
C\alpha\int_{\tau_j}^t\int_{\partial\Omega(x,r_j)} a(y,s)z_\epsilon^\alpha\zeta_j^2\,d\sigma ds &\le C\alpha\|a\|_{L^\infty(Q_0)}\int_{\tau_j}^t\int_{\partial\Omega_j} z_\epsilon^\alpha\zeta_j^2\,d\sigma ds\\
&\le\frac{1}{2}\iint_{Q_j}|\nabla[z_\epsilon^{\frac{\alpha}{2}}\zeta_j]|^2\,dyds+C\alpha^2\|a\|_{L^\infty(Q_0)}^2\iint_{Q_j} z_\epsilon^\alpha\zeta_j^2\,dyds.
\end{aligned}
\tag{3.9}
$$

We deduce from (3.6), (3.8) and (3.9) that

$$
\begin{aligned}
&\sup_{\tau_j<s<t}\int_{\Omega(x,r_j)} z_\epsilon^\alpha\zeta_j^2\,dy+\iint_{Q_j}|\nabla[z_\epsilon^{\frac{\alpha}{2}}\zeta_j]|^2\,dyds\\
&\le C\left[\alpha^2\|a\|_{L^\infty(Q_0)}^2+\frac{2^{2j}}{(R_2-R_1)^2}+\frac{2^j}{t_1-t_2}\right]\iint_{Q_j} z_\epsilon^\alpha\,dyds
\end{aligned}
\tag{3.10}
$$

for all $j=0,1,2,\dots$ and $\alpha\ge\alpha_0$. This together with (1.4) implies that

$$
\begin{aligned}
&\left(\iint_{Q_{j+1}} z_\epsilon^{\kappa\alpha}\,dyds\right)^{1/\kappa}\\
&\le C\left[\alpha^2\|a\|_{L^\infty(Q_0)}^2+\frac{2^{2j}}{(R_2-R_1)^2}+\frac{2^j}{t_1-t_2}\right]\iint_{Q_j} z_\epsilon^\alpha\,dyds
\end{aligned}
\tag{3.11}
$$

for all $j=0,1,2,\dots$ and $\alpha\ge\alpha_0$, where $\kappa:=1+2/N$. Furthermore, by (3.10) with $\alpha=2$ we have (3.5).

We prove (3.4) in the case $r\ge 2$. Setting

$$
I_j:=\|z_\epsilon\|_{L^{\alpha_j}(Q_j)},\qquad \alpha_j:=r\kappa^j,
$$

by (3.11) we have

$$
I_{j+1}\le C^{\frac{1}{\alpha_j}}\left[\alpha_j^2\|a\|_{L^\infty(Q_0)}^2+\frac{2^{2j}}{(R_2-R_1)^2}+\frac{2^j}{t_1-t_2}\right]^{\frac{1}{\alpha_j}} I_j\le C^{\frac{j}{\alpha_j}}(CD)^{\frac{1}{\alpha_j}}I_j \tag{3.12}
$$

for all $j=0,1,2,\dots$, where $D:=\|a\|_{L^\infty(Q_0)}^2+(R_2-R_1)^{-2}+(t_1-t_2)^{-1}$. Since

$$
\sum_{j=0}^\infty\frac{1}{\alpha_j}=\frac{1}{r}\sum_{j=0}^\infty\kappa^{-j}=\frac{1}{r(1-\kappa^{-1})}=\frac{N+2}{2r},\qquad \sum_{j=0}^\infty\frac{j}{\alpha_j}<\infty,
$$

we deduce from (3.12) that

$$\|z_\epsilon\|_{L^\infty(Q_\infty)} = \lim_{j\to\infty} I_j \le C^{\sum_{j=0}^\infty \frac{j}{\alpha_j}} (CD)^{\sum_{j=0}^\infty \frac{1}{\alpha_j}} I_0 \le CD^{(N+2)/2r}\|z_\epsilon\|_{L^r(Q_0)},$$

which implies

$$\|z_\epsilon\|_{L^\infty(\Omega(x,R_1)\times(t_1,t))} \le CD^{\frac{N+2}{2r}}\left(\int_{t_2}^t\int_{\Omega(x,R_2)} z_\epsilon^r\, dyds\right)^{1/r}, \tag{3.13}$$

where $r \ge 2$. Then, passing the limit as $\epsilon \to 0$, we obtain (3.4).

On the other hand, for the case $1 \le r < 2$, applying (3.13) with $r = 2$ to the cylinders Q_j and Q_{j+1}, we have

$$\begin{aligned}
\|z_\epsilon\|_{L^\infty(Q_{j+1})} &\le C\left((2^{2j}D)^{\frac{N+2}{2}}\iint_{Q_j} z_\epsilon^2\, dyds\right)^{\frac{1}{2}} \\
&\le Cb^j\|z_\epsilon\|^{1-r/2}_{L^\infty(Q_j)}\left(D^{(N+2)/2}\iint_{Q_j} z_\epsilon^r\, dyds\right)^{\frac{1}{2}},
\end{aligned}$$

where $b = 2^{(N+2)/2}$. Then, for any $\nu > 0$, we have

$$\begin{aligned}
\|z_\epsilon\|_{L^\infty(Q_{j+1})} &\le \nu\|z_\epsilon\|_{L^\infty(Q_j)} + C\nu^{-\frac{2-r}{r}} b^{\frac{2}{r}j} D^{\frac{N+2}{2r}}\left(\iint_{Q_j} z_\epsilon^r\, dyds\right)^{1/r} \\
&\le \nu^{j+1}\|z_\epsilon\|_{L^\infty(Q_0)} + C\nu^{-\frac{2-r}{r}}\sum_{i=0}^{j}(\nu b^{\frac{2}{r}})^i D^{\frac{N+2}{2r}}\left(\iint_{Q_0} z_\epsilon^r\, dyds\right)^{1/r}
\end{aligned}$$

for $j = 1, 2, \ldots$. Taking a sufficiently small ν if necessary, we see that

$$\|z_\epsilon\|_{L^\infty(Q_{j+1})} \le \nu^{j+1}\|z_\epsilon\|_{L^\infty(Q_0)} + CD^{\frac{N+2}{2r}}\left(\iint_{Q_0} z_\epsilon^r\, dyds\right)^{1/r}$$

for $j = 1, 2, \ldots$. Passing to the limit as $j \to \infty$ and $\epsilon \to 0$, we obtain

$$\|z_0\|_{L^\infty(Q_\infty)} \le CD^{\frac{N+2}{2r}}\left(\iint_{Q_0} z_0^r\, dyds\right)^{1/r},$$

which implies (3.4) in the case $1 \le r < 2$. Thus Lemma 3.1 follows. □

Lemma 3.2 *Assume the same conditions as in Theorem* 1.1. *Let r satisfy* (1.6) *and $r > 1$. Let v be a $L^r_{uloc}(\Omega)$-solution of* (1.1) *in $\Omega\times[0,T]$, where $T > 0$. Then there exists a positive constant Λ such that, if*

$$\rho^{\frac{r}{p-1}-N}\Psi_{r,\rho}[v](T) \le \Lambda \tag{3.14}$$

for some $\rho \in (0, \rho_*/2)$, then

$$\Psi_{r,\rho}[v](t) \le 5M\Psi_{r,\rho}[v](\tau), \tag{3.15}$$

$$\sup_{x\in\overline{\Omega}}\int_\tau^t\int_{\partial\Omega(x,\rho)} |v|^{p+r-1}\, d\sigma ds \le C\Lambda^{\frac{p-1}{r}}\Psi_{r,\rho}[v](\tau), \tag{3.16}$$

for all $0 \le \tau \le t \le T$ *with* $t - \tau \le \mu\rho^2$, *where* C *and* μ *are positive constants depending only on* N, Ω, p *and* r.

Proof. Let $x \in \overline{\Omega}$ and let ζ and k be as in Lemma 2.4. By (3.14) we can take a sufficiently small $\epsilon > 0$ so that

$$\rho^{\frac{r}{p-1}-N}\Psi_{r,\rho}[v_\epsilon](T) \le 2\Lambda, \tag{3.17}$$

where $v_\epsilon := \max\{\pm v, 0\} + \epsilon$. Similarly to (3.8), for any $0 < \tau < t \le T$, multiplying (1.1) by $v_\epsilon^{r-1}\zeta^k$ and integrating it in $\Omega \times (\tau, t)$, we obtain

$$\begin{aligned} &\int_{\Omega(x,2\rho)} v_\epsilon(y,s)^r \zeta^k \, dy \bigg|_{s=\tau}^{s=t} + \int_\tau^t \int_{\Omega(x,\rho)} |\nabla v_\epsilon^{\frac{r}{2}}|^2 \, dyds \\ &\le C\rho^{-2} \int_\tau^t \int_{\Omega(x,2\rho)} v_\epsilon^r \, dyds + C \int_\tau^t \int_{\partial\Omega(x,2\rho)} v_\epsilon^{p+r-1}\zeta^k \, d\sigma ds. \end{aligned} \tag{3.18}$$

This together with $v \in C(\overline{\Omega} \times [\tau, T]) \cap L^\infty(\tau, T : L^\infty(\Omega))$ (see Definition 1.1) implies that

$$\sup_{x \in \overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla v_\epsilon^{\frac{r}{2}}|^2 \, dyds < \infty. \tag{3.19}$$

Furthermore, by Lemma 2.4, (3.17) and (3.18) we have

$$\begin{aligned} &\int_{\Omega(x,2\rho)} v_\epsilon(y,s)^r \zeta^k \, dy \bigg|_{s=\tau}^{s=t} + \int_\tau^t \int_{\Omega(x,\rho)} |\nabla v_\epsilon^{\frac{r}{2}}|^2 \, dyds \le C\rho^{-2} \int_\tau^t \int_{\Omega(x,2\rho)} v_\epsilon^r \, dyds \\ &\quad + C(2\Lambda)^{\frac{p-1}{r}} \left[\sup_{x \in \overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla v_\epsilon^{\frac{r}{2}}|^2 \, dyds + \rho^{-2}(t-\tau)\Psi_{r,\rho}[v_\epsilon](t) \right] \end{aligned} \tag{3.20}$$

for $0 < \tau < t \le T$. Therefore, by Lemma 2.1, (1.17) and (3.20) we obtain

$$\begin{aligned} &\sup_{x \in \overline{\Omega}} \int_{\Omega(x,2\rho)} v_\epsilon(y,t)^r \, dy + \sup_{x \in \overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla v_\epsilon^{\frac{r}{2}}|^2 \, dyds \\ &\le M \sup_{x \in \overline{\Omega}} \int_{\Omega(x,\rho)} v_\epsilon(y,\tau)^r \, dy + C\rho^{-2}(t-\tau)\Psi_{r,\rho}[v_\epsilon](t) \\ &\quad + C(2\Lambda)^{\frac{p-1}{r}} \left[\sup_{x \in \overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla v_\epsilon^{\frac{r}{2}}|^2 \, dyds + \rho^{-2}(t-\tau)\Psi_{r,\rho}[v_\epsilon](t) \right] \end{aligned} \tag{3.21}$$

for $0 < \tau < t \le T$. Taking a sufficiently small Λ if necessary, we deduce from (3.19) and (3.21) that

$$\begin{aligned} &\sup_{x \in \overline{\Omega}} \int_{\Omega(x,\rho)} v_\epsilon(y,t)^r \, dy + \frac{1}{2} \sup_{x \in \overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla v_\epsilon^{\frac{r}{2}}|^2 \, dyds \\ &\quad \le M \sup_{x \in \overline{\Omega}} \int_{\Omega(x,\rho)} v_\epsilon(y,\tau)^r \, dy + C\rho^{-2}(t-\tau)\Psi_{r,\rho}[v_\epsilon](t). \end{aligned}$$

Taking a sufficiently small $\mu \in (0,1]$, we obtain

$$
\begin{aligned}
&\Psi_{r,\rho}[v_\epsilon](t) + \frac{1}{2}\sup_{x\in\overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla v_\epsilon^{\frac{r}{2}}|^2 \, dyds \\
&\le 2M\Psi_{r,\rho}[v_\epsilon](\tau) + C\rho^{-2}(t-\tau)\Psi_{r,\rho}[v_\epsilon](t) \le 2M\Psi_{r,\rho}[v_\epsilon](\tau) + \frac{1}{2}\Psi_{r,\rho}[v_\epsilon](t)
\end{aligned}
\tag{3.22}
$$

for $0 < \tau < t \le T$ with $t - \tau \le \mu\rho^2$. This implies that

$$
\Psi_{r,\rho}[\max\{\pm v, 0\}](t) \le \Psi_{r,\rho}[v_\epsilon](t) \le 4M\Psi_{r,\rho}[v_\epsilon](\tau) \le 5M\Psi_{r,\rho}[v](\tau) + C\epsilon^r\rho^N \tag{3.23}
$$

for $0 < \tau < t \le T$ with $t - \tau \le \mu\rho^2$. Furthermore, by Lemma 2.4, (3.22) and (3.23) we have

$$
\begin{aligned}
\int_\tau^t \int_{\partial\Omega(x,\rho)} \max\{\pm v, 0\}^{p+r-1} \, d\sigma ds &\le \int_\tau^t \int_{\partial\Omega(x,\rho)} v_\epsilon^{p+r-1} \, d\sigma ds \\
&\le C\Lambda^{\frac{p-1}{r}}\Psi_{r,\rho}[v_\epsilon](\tau) \le C\Lambda^{\frac{p-1}{r}}\Psi_{r,\rho}[v](\tau) + C\epsilon^r\rho^N.
\end{aligned}
\tag{3.24}
$$

Since τ and ϵ is arbitrary, by (3.23) and (3.24) we obtain (3.15) and (3.16). Thus Lemma 3.2 follows. □

Lemma 3.3 *Assume the same conditions as in Lemma* 3.1. *Let* r *satisfy* (1.6) *and* $r > 1$. *Then there exists a positive constant* Λ *such that, if*

$$
\rho^{\frac{r}{p-1}-N}\left(\Psi_{r,\rho}[v](T) + \Psi_{r,\rho}[w](T)\right) \le \Lambda \tag{3.25}
$$

for some $\rho \in (0, \rho_*/2)$, *then*

$$
\Psi_{r,\rho}[z_0](t) \le C\Psi_{r,\rho}[z_0](\tau) \tag{3.26}
$$

for $0 \le \tau < t \le T$ *with* $t - \tau \le \mu\rho^2$, *where* C *and* μ *are positive constants depending only on* N, Ω, p *and* r.

Proof. Let $x \in \overline{\Omega}$ and ζ be as in Lemma 2.4. Let k be as in Lemma 2.4 and $\epsilon > 0$. Similarly to (3.18), we have

$$
\begin{aligned}
&\int_{\Omega(x,2\rho)} z_\epsilon(y,s)^r\zeta^k \, dy\Big|_{s=\tau}^{s=t} + \int_\tau^t \int_{\Omega(x,2\rho)} |\nabla z_\epsilon^{\frac{r}{2}}|^2\zeta^k \, dyds \\
&\le C\rho^{-2}\int_\tau^t \int_{\Omega(x,2\rho)} z_\epsilon^r \, dyds + C\int_\tau^t \int_{\partial\Omega(x,2\rho)} a(y,s)z_\epsilon^r\zeta^k \, d\sigma ds
\end{aligned}
\tag{3.27}
$$

for all $0 < \tau < t \le T$. This together with z_ϵ, $a \in C(\overline{\Omega} \times [\tau, T]) \cap L^\infty(\Omega \times (\tau, T))$ implies that

$$
\sup_{x\in\overline{\Omega}} \int_\tau^t \int_{\Omega(x,2\rho)} |\nabla z_\epsilon^{\frac{r}{2}}|^2 \, dyds < \infty \tag{3.28}
$$

for $0<\tau<t\le T$. On the other hand, by the Hölder inequality and (3.3) we have

$$
\begin{aligned}
\int_\tau^t\int_{\partial\Omega(x,2\rho)} a(y,\tau)z_\epsilon^r\zeta^k\,d\sigma ds \le C\left(\int_\tau^t\int_{\partial\Omega(x,2\rho)}(|v|^{p+r-1}+|w|^{p+r-1})\,d\sigma ds\right)^{\frac{p-1}{p+r-1}}\\
\times\left(\int_\tau^t\int_{\partial\Omega(x,2\rho)} z_\epsilon^{p+r-1}\zeta^k\,d\sigma ds\right)^{\frac{r}{p+r-1}}.
\end{aligned}
\tag{3.29}
$$

Let Λ and μ be sufficiently small positive constants. Then, by Lemma 2.1, (3.16) and (3.25) we see that

$$
\begin{aligned}
&\int_\tau^t\int_{\partial\Omega(x,2\rho)}(|v|^{p+r-1}+|w|^{p+r-1})\,d\sigma ds\\
&\le M\sup_{x\in\overline{\Omega}}\int_\tau^t\int_{\partial\Omega(x,\rho)}(|v|^{p+r-1}+|w|^{p+r-1})\,d\sigma ds\\
&\le C\Lambda^{\frac{p-1}{r}}\left\{\Psi_{r,\rho}[v](\tau)+\Psi_{r,\rho}[w](\tau)\right\}\le C\Lambda^{\frac{p+r-1}{r}}\rho^{-\frac{r}{p-1}+N}
\end{aligned}
\tag{3.30}
$$

for all $0<\tau<t\le T$ with $t-\tau\le\mu\rho^2$. Similarly, by Lemma 2.4 we obtain

$$
\begin{aligned}
\int_\tau^t\int_{\partial\Omega(x,2\rho)} z_\epsilon^{p+r-1}\zeta^k\,d\sigma ds \le C\left(\rho^{\frac{r}{p-1}-N}\Psi_{r,\rho}[z_\epsilon](t)\right)^{\frac{p-1}{r}}\\
\times\left[\sup_{x\in\overline{\Omega}}\int_\tau^t\int_{\Omega(x,\rho)}|\nabla(z_\epsilon)^{\frac{r}{2}}|^2\,dyds+\rho^{-2}(t-\tau)\Psi_{r,\rho}[z_\epsilon](\tau)\right]
\end{aligned}
\tag{3.31}
$$

for all $0<\tau<t\le T$ with $t-\tau\le\mu\rho^2$. Then we deduce from (3.29)–(3.31) that

$$
\begin{aligned}
&\int_\tau^t\int_{\partial\Omega(x,2\rho)} a(y,t)z_\epsilon^r\zeta^k\,d\sigma ds\\
&\le C\Lambda^{\frac{p-1}{r}}\left(\Psi_{r,\rho}[z_\epsilon](t)\right)^{\frac{p-1}{p+r-1}}\\
&\quad\times\left[\sup_{x\in\overline{\Omega}}\int_\tau^t\int_{\Omega(x,\rho)}|\nabla(z_\epsilon)^{\frac{r}{2}}|^2\,dyds+\rho^{-2}(t-\tau)\Psi_{r,\rho}[z_\epsilon](t)\right]^{\frac{r}{p+r-1}}\\
&\le C\Lambda^{\frac{p-1}{r}}\left[\sup_{x\in\overline{\Omega}}\int_\tau^t\int_{\Omega(x,\rho)}|\nabla z_\epsilon^{\frac{r}{2}}|^2\,dyds+\Psi_{r,\rho}[z_\epsilon](t)+\rho^{-2}(t-\tau)\Psi_{r,\rho}[z_\epsilon](\tau)\right]
\end{aligned}
\tag{3.32}
$$

for all $0<\tau<t\le T$ with $t-\tau\le\mu\rho^2$. Therefore, by Lemma 2.1, (3.27) and (3.32) we have

$$
\begin{aligned}
&\sup_{x\in\overline{\Omega}}\int_{\Omega(x,\rho)} z_\epsilon^r\,dy+\sup_{x\in\overline{\Omega}}\int_\tau^t\int_{\Omega(x,\rho)}|\nabla z_\epsilon^{\frac{r}{2}}|^2\,dyds\\
&\quad\le M\Psi_{r,\rho}[z_\epsilon](\tau)+C\rho^{-2}(t-\tau)\Psi_{r,\rho}[z_\epsilon](t)\\
&\qquad+C\Lambda^{\frac{p-1}{r}}\left[\sup_{x\in\overline{\Omega}}\int_\tau^t\int_{\Omega(x,\rho)}|\nabla z_\epsilon^{\frac{r}{2}}|^2\,dyds+\Psi_{r,\rho}[z_\epsilon](t)+\rho^{-2}(t-\tau)\Psi_{r,\rho}[z_\epsilon](\tau)\right]
\end{aligned}
$$

for all $0 < \tau < t \le T$ with $t - \tau \le \mu\rho^2$. Then, taking sufficiently small constants Λ and μ if necessary, we obtain

$$\Psi_{r,\rho}[z_\epsilon](t) \le 4M\Psi_{r,\rho}[z_\epsilon](\tau)$$

for all $0 < \tau < t \le T$ with $t - \tau \le \mu\rho^2$. This implies (3.26), and the proof is complete. □

Now we are ready to complete the proof of Theorems 1.1 and 1.2 in the case $r > 1$.

Proof of Theorem 1.1 in the case $r > 1$. Let γ_1 be a sufficiently small positive constant and assume (1.7). Let $\{\varphi_n\}$ satisfy (1.13) and define T_n^* and T_n^{**} as in (1.17). Then it follows from (1.16) that

$$\rho^{\frac{r}{p-1}-N}\Psi_{r,\rho}[u_n](t) \le 6M\rho^{\frac{r}{p-1}-N}\Psi_{r,\rho}[u_n](0) \le 6M(2\gamma_1)^r \tag{3.33}$$

for all $0 \le t \le T_n^*$. Taking a sufficiently small γ_1 if necessary, by Lemma 3.2, (1.16) and (3.33), we can find a constant $\mu > 0$ such that

$$\Psi_{r,\rho}[u_n](t) \le 5M\Psi_{r,\rho}[u_n](0) < 6M\Psi_{r,\rho}[u_n](0) \le C\|\varphi\|_{r,\rho}^r \tag{3.34}$$

for $0 \le t \le \min\{T_n^*, \mu\rho^2\}$. On the other hand, we apply Lemma 3.1 with $R_1 = \rho/2$, $R_2 = \rho$, $t_1 = t/2$ and $t_2 = t/4$ to obtain

$$\|u_n(t)\|_{L^\infty(\Omega(x,\rho/2))} \le CD^{\frac{N+2}{2r}}\left(\int_{t/4}^t\int_{\Omega(x,\rho)} |u_n|^r\,dyds\right)^{1/r}, \tag{3.35}$$

$$\int_{t/2}^t\int_{\Omega(x,\rho/2)} |\nabla u_n|^2\,dyds \le CD\int_{t/4}^t\int_{\Omega(x,\rho)} |u_n|^2\,dyds, \tag{3.36}$$

for all $x \in \overline{\Omega}$ and $t \in (0, T_n)$. where $D = \||u_n|^{p-1}\|^2_{L^\infty(\Omega(x,\rho)\times(t/4,t))} + \rho^{-2} + t^{-1}$. By (1.17), (3.34) and (3.35) we have

$$\|u_n(t)\|_{L^\infty(\Omega)} \le Ct^{-\frac{N}{2r}}\|\varphi\|_{r,\rho} \le C\gamma_1 t^{-\frac{1}{2(p-1)}}(\rho^{-2}t)^{-\frac{N}{2r}+\frac{1}{2(p-1)}}, \tag{3.37}$$

$$\sup_{x\in\overline{\Omega}}\int_{t/2}^t\int_{\Omega(x,\rho)} |\nabla u_n|^2\,dyds \le C\rho^N\|u_n\|^2_{L^\infty(\Omega\times(t/4,t))} \le C\rho^N t^{-\frac{N}{r}}\|\varphi\|^2_{r,\rho}, \tag{3.38}$$

for all $0 < t \le \min\{\mu\rho^2, T_n^*, T_n^{**}\}$. Since $r \ge N(p-1)$, taking sufficiently small $\gamma_1 > 0$ and $\mu > 0$ if necessary, by (3.37) we have

$$(\rho^{-2}t)^{\frac{1}{2}} + t^{\frac{1}{2}}\|u_n(t)\|^{\frac{p-1}{2}}_{L^\infty(\Omega)} \le \mu^{\frac{1}{2}} + (C\gamma_1)^{p-1}\mu^{-\frac{N(p-1)}{2r}+\frac{1}{2}} \le 1$$

for $0 < t \le \min\{\mu\rho^2, T_n^*, T_n^{**}\}$. This implies that $T_n > T_n^{**} > \min\{T_n^*, \mu\rho^2\}$ for $n = 1, 2, \ldots$. Then, by (3.34) we see that $T_n^* > \mu\rho^2$ for $n = 1, 2, \ldots$. Therefore, by (3.34), (3.37) and (3.38) we obtain

$$\|u_n(t)\|_{L^\infty(\Omega)} \le Ct^{-\frac{N}{2r}}\|\varphi\|_{r,\rho}, \tag{3.39}$$

$$\sup_{x\in\overline{\Omega}}\int_{t/2}^t\int_{\Omega(x,\rho)} |\nabla u_n|^2\,dyds \le C\rho^N t^{-\frac{N}{r}}\|\varphi\|^2_{r,\rho}, \tag{3.40}$$

$$\sup_{0<t<\mu\rho^2}\|u_n(t)\|_{r,\rho} \le C\|\varphi\|_{r,\rho}, \tag{3.41}$$

for $0 < t \le \mu\rho^2$ and $n = 1, 2, \ldots$.

Applying [12, Theorem 6.2] with the aid of (3.39), we see that u_n $(n = 1, 2, \ldots)$ are uniformly bounded and equicontinuous on $K \times [\tau, \mu\rho^2]$ for any compact set $K \subset \overline{\Omega}$ and $\tau \in (0, \mu\rho^2]$. Then, by the Ascoli-Arzelà theorem and the diagonal argument we can find a subsequence $\{u_{n'}\}$ and a continuous function u in $\Omega \times (0, \mu\rho^2]$ such that

$$\lim_{n'\to\infty} \|u_{n'} - u\|_{L^\infty(K\times[\tau,\mu\rho^2])} = 0$$

for any compact set $K \subset \overline{\Omega}$ and $\tau \in (0, \mu\rho^2]$. This together with (3.39) and (3.41) implies (1.8) and (1.9). Furthermore, by (3.40), taking a subsequence if necessary, we see that

$$\lim_{n'\to\infty} u_{n'} = u \quad \text{weakly in } L^2([\tau, \mu\rho^2] : W^{1,2}(\Omega \cap B(0, R)))$$

for any $R > 0$ and $0 < \tau < \mu\rho^2$. This implies that u satisfies (1.5).

On the other hand, since u_n is a $L^r_{uloc}(\Omega)$-solution of (1.1) (see (1.15)), we see that

$$u_n \in C([0, \mu\rho^2] : L^r_{uloc,\rho}(\Omega)).$$

Furthermore, by Lemma 3.3 and (3.33), taking a sufficiently small γ_1 if necessary, we have

$$\sup_{0<\tau<\mu\rho^2} \|u_m(\tau) - u_n(\tau)\|_{r,\rho} \le C\|u_m(0) - u_n(0)\|_{r,\rho}, \quad m, n = 1, 2, \ldots.$$

This means that $\{u_n\}$ is a Cauchy sequence in $C([0, \mu\rho^2] : L^r_{uloc,\rho}(\Omega))$, which implies

$$u \in C([0, \mu\rho^2] : L^r_{uloc,\rho}(\Omega)). \tag{3.42}$$

Therefore we see that u is a $L^r_{uloc}(\Omega)$-solution of (1.1) in $\Omega \times [0, \mu\rho^2]$ satisfying (1.8) and (1.9), and the proof of Theorem 1.1 for the case $r > 1$ is complete. □

Proof of Theorem 1.2 in the case $r > 1$. Let v and w be $L^r_{uloc}(\Omega)$-solutions of (1.1) in $\Omega \times [0, T)$, where $T > 0$. Let γ_2 be a sufficiently small constant and assume (1.11). We can assume, without loss of generality, that $\rho \in (0, \rho_*/2)$. Since v, $w \in C([0, T] : L^r_{uloc,\rho}(\Omega))$, we can find a constant $T' \in (0, T)$ such that

$$\rho^{\frac{1}{p-1} - \frac{N}{r}} \left[\sup_{0<\tau\le T'} \|v(\tau)\|_{r,\rho} + \sup_{0<\tau\le T'} \|w(\tau)\|_{r,\rho} \right] \le 2\gamma_2. \tag{3.43}$$

Furthermore, for any $T'' \in (T', T)$, since v, $w \in L^\infty(\Omega \times (T', T''))$, we see that

$$\tilde\rho^{\frac{1}{p-1} - \frac{N}{r}} \left[\sup_{T'<\tau\le T''} \|v(\tau)\|_{r,\tilde\rho} + \sup_{T'<\tau\le T''} \|w(\tau)\|_{r,\tilde\rho} \right] \le \gamma_2 \tag{3.44}$$

for some $\tilde\rho \in (0, \rho)$. Since $v(x, 0) \le w(x, 0)$ for almost all $x \in \Omega$, by (3.43) and (3.44) we apply Lemma 3.3 to obtain

$$\sup_{0<\tau<\min\{\mu\tilde\rho^2, T''\}} \|(v(\tau) - w(\tau))_+\|_{r,\tilde\rho} \le C\|(v(0) - w(0))_+\|_{r,\tilde\rho} = 0$$

for some constant $\mu > 0$. This implies that $v(x, t) \le w(x, t)$ in $\Omega \times (0, \min\{\mu\tilde\rho^2, T''\}]$. Repeating this argument, we see that $v(x, t) \le w(x, t)$ in $\Omega \times (0, T'']$. Finally, since T'' is arbitrary, we see that $v(x, t) \le w(x, t)$ in $\Omega \times (0, T)$, and the proof is complete. □

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